Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 25, 26

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Textbook Sections: 11.10

Todays Topic: Taylor Series

A quick recap of last week:

- **Def:** $\sum_{n=0}^{\infty} c_n (x-a)^n$ where $c_n \in \mathbb{R}$, $a \in \mathbb{R}$ are fixed, and x is a variable
  is a **power series**.

- **Key Fact:** $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ is a nice function when x is in a **domain of convergence**, i.e., $\exists R \geq 0$ (the **radius of convergence**) so that $f(x)$ converges when $|x-a| < R$ and diverges when $|x-a| > R$.
  ⚠ $f(x)$ could converge / diverge at $x = a \pm R$ (the endpoints of the domain)

  Key Tools: Ratio and Root Tests

- When thinking of $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ as a function, we would like to use the techniques of calculus:

Key Result We can differentiate / integrate $f(x)$ by doing these operations "term-by-term" in the power series, preserving the radius of convergence.

Todays Goal: Given a function $f(x)$ we want to find a power series representation. There are some problems, however:

1) The techniques we touched on in the last class required $f(x)$ to somehow resemble a function that we already have a series representation for. What do we do for more complicated functions?

2) A more serious problem: We never addressed what functions can actually be represented as power series to begin with. As it turns out, this can get quite tricky to sort out.

So, today, we'll focus on a narrower question which is "well-posed:"

Challenge: Suppose $f(x)$ has a power series representation

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

on $|x-a| < R$. Can we tell what the coefficients c_n must be?

Step 1: If we evaluate @ $x=a$, we get

$$f(a) = c_0 + c_1(a-a) + c_2(a-a)^2 + \cdots$$
$$= c_0$$

So we at least know $c_0 = f(a)$.

Step 2: Let's use our term-by-term differentiation theorem, which is valid in the domain $|x-a| < R$:

$$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1} = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

Now let's plug in $x=a$ again:

$$f'(a) = c_1 + 2c_2(a-a) + 3c_3(a-a)^2 + \cdots$$
$$= c_1$$

So now we have figured out $c_1 = f'(a)$.

Step 3-∞: Let's keep repeating this process:

- $f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2} \rightsquigarrow f''(a) = 2 \cdot c_2 \rightsquigarrow c_2 = \frac{f''(a)}{2}$
- $f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) c_n (x-a)^{n-3} \rightsquigarrow f'''(a) = 3 \cdot 2 \cdot 1 \cdot c_3 \rightsquigarrow c_3 = \frac{f'''(a)}{3!}$
- $\vdots$
- $f^{(N)}(x) = \sum_{n=N}^{\infty} n(n-1)(n-2)\cdots(n-N+1) c_n (x-a)^{n-N}$

$$\rightsquigarrow f^{(N)}(a) = N \cdot (N-1)(N-2) \cdots 2 \cdot 1 \cdot c_N \rightsquigarrow \boxed{c_N = \frac{f^{(N)}(a)}{N!}}$$

So, we've just proved:

Theorem: If $f(x)$ has a power series representation on $|x-a|<R$:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

then we must have

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

We call

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

the Taylor Series of $f(x)$ at a.

N.B. Some weird people still call a Taylor series centered at $a=0$ a Maclaurin series.

Examples: Let's assume for the moment that e^x and $\sin x$ are representable as power series, i.e., they are equal to their Taylor series (we'll prove this in a second, or in the next class...)
What are their Taylor series? Let's fix $a=0$ for clarity:

1) Given that $e^x = \sum_{n=0}^{\infty} c_n x^n$, we have

$$c_n = \frac{(e^x)^{(n)}(0)}{n!} = \frac{e^x|_{x=0}}{n!} = \frac{1}{n!}$$

Thus we recover our old friend $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

2) Given that $\sin x = \sum_{n=0}^{\infty} c_n x^n$, we have

$$c_n = \frac{(\sin x)^{(n)}(0)}{n!} = \begin{cases} \frac{\sin(0)}{0!} = 0 \\ \frac{\cos(0)}{1!} = \frac{1}{1!} \\ \frac{-\sin(0)}{2!} = 0 \\ \frac{-\cos(0)}{3!} = -\frac{1}{3!} \\ \frac{\sin(0)}{4!} = 0 \\ \vdots \text{ repeating in cycles of length 4.} \end{cases}$$

$$\rightsquigarrow \sin(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!}$$

⚠ Check using a ratio/root test that these Taylor series converge on all of $(-\infty, \infty)$, the same domain as e^x and $\sin(x)$. It's remarkable that all of the derivatives of these functions at just one single point can describe the function everywhere else on $(-\infty, \infty)$.

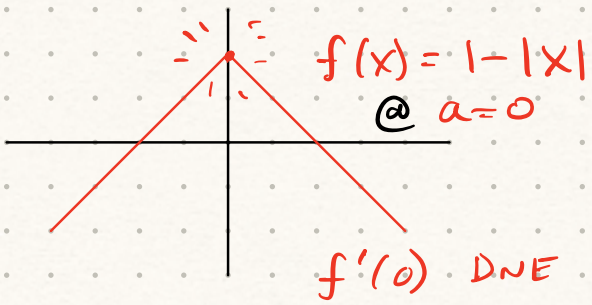
Here are some others to try after class / before next time:

3) $f(x) = 5x^3 + 3x + 7$ at $a=0$ and then at $a=2$
4) $f(x) = \cosh(x)$ at $a=0$
5) $f(x) = \sqrt{x}$ at $a=1$

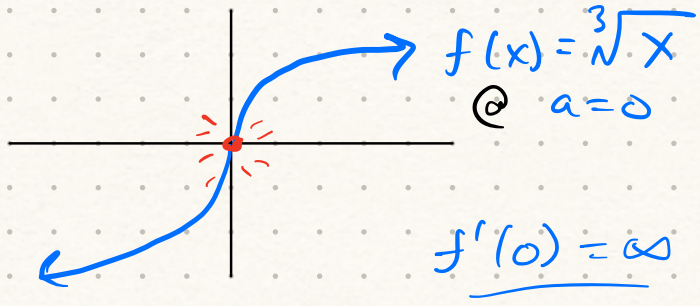
Let's now address the "more serious" problem we acknowledged last time:

Q: How do we know if some function $f(x)$ has a Taylor series representation to begin with?

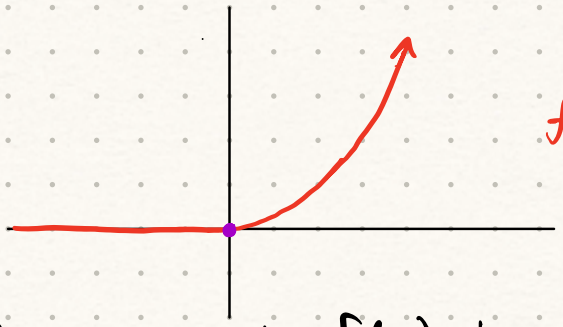
Preliminary Answer: One thing we know for sure: $f(x)$ needs to have infinitely many derivatives (at least at $x=a$). Functions that fail this test include things like:



or



⚠ But be careful! It's not enough to just have ∞-many derivatives at $x=a$ to have a Taylor series representation in an "open neighborhood" of $x=a$. A very sneaky example is the following:



$$f(x) = \begin{cases} 0 & x \le 0 \\ e^{-\frac{1}{x^2}} & x > 0 \end{cases}$$

↖ A "zombie" function which comes back to life!

It turns out $f(x)$ has ∞-many derivatives <u>everywhere</u> including at $x=0$.

Intuitively, this is because $e^{-\frac{1}{x^2}} = \frac{1}{e^{1/x^2}} \to 0$ extremely quickly as $x \searrow 0$, as do all of its derivatives, eg:

$$\frac{d}{dx}\left(e^{-\frac{1}{x^2}}\right) = \frac{2}{x^3}e^{-\frac{1}{x^2}} = \frac{2}{x^3 e^{\frac{1}{x^2}}} \to 0 \quad \text{as } x \searrow 0$$

$$\frac{d^2}{dx^2}\left(e^{-\frac{1}{x^2}}\right) = \left(\frac{2}{x^3}\right)^2 e^{-\frac{1}{x^2}} - \frac{6}{x^4}e^{-\frac{1}{x^2}} = \left(\left(\frac{2}{x^3}\right)^2 - \frac{6}{x^4}\right)\frac{1}{e^{\frac{1}{x^2}}} \to 0 \quad \text{as } x \searrow 0$$

$\vdots$

etc.

Thus, $f(x)$ is smooth, and @ $x=0$, $f^{(n)}(0) = 0$ for all n.

Thus, if $f(x)$ we representable as a Taylor series with $a=0$, then we would need to have $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = 0$

in a neighborhood of $x=0$, even though $f(x)$ is definitely <u>not</u> 0 true!

<u>In other words:</u> There are smooth functions (i.e. they have ∞-many derivatives everywhere) which are <u>not</u> analytic (i.e. they do <u>not</u> have a power series representation)!

<u>Let's focus on understanding when a function $f(x)$ does have a power series representation:</u>

First of all, if $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ for $|x-a| < R$, then by definition

$$f(x) = \lim_{N\to\infty} T_N(x) := \lim_{N\to\infty} \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

In other words, $f(x)$ is the limit of the "partial sum functions" $T_N(x)$.

We call $T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$ the n^{th} degree Taylor polynomial.

Now, as we did for series of real numbers, we define the remainder, which is now a function of x:

$$R_N(x) = f(x) - T_N(x) = f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

If $f(x)$ is smooth, then $R_N(x)$ has to be as well, since $T_N(x)$ is just a polynomial. The interesting question is therefore:

Key Question: As $N \to \infty$, does $|R_N(x)| \to 0$ for all $|x-a|<R$?

This is what it takes to have $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ on $|x-a|<R$.

Key Tool: To understand when $|R_N(x)| \to 0$, the following are often useful: In these results, we only need to assume $f^{(n+1)}(x)$ is continuous:

Theorem: (Taylor's Inequality)

If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le R$, then

$$|R_N(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{on } |x-a| \le R$$

Theorem: (Integral form of the remainder)

If $f^{(n+1)}(x)$ is continuous on $|x-a| \le R$, then

$$R_N(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt \quad \text{on } |x-a| \le R.$$

Theorem: (Leibniz form of the remainder)

If $f^{(n+1)}(x)$ is continuous on $|x-a| \le R$, then for each such x, there exists a z between a and x so that

$$R_N(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}.$$

For the proofs, see the book / search online. The basic idea is to use the FTC and MVT.

Examples:

6) e^x is analytic.

We need to show that $R_N(x) = e^x - \sum_{n=0}^{N} \frac{1}{n!} x^n$ has $|R_N(x)| \to 0$ for all $x \in \mathbb{R}$ as $N \to \infty$.

(i) Notice that $\frac{d^n}{dx^n} e^x = e^x$. So, if $|x| \le R$ (for any fixed $R > 0$), $\left|\frac{d^n}{dx^n} e^x\right| = |e^x| = e^x \le e^R$ for every $n \ge 1$.

(ii) By Taylor's Inequality, for any $|x| \le R$

$$0 \le |R_N(x)| \le \frac{e^R}{(N+1)!} |x|^{N+1} = \frac{e^R \cdot R^{N+1}}{(N+1)!} \to 0 \text{ as } N \to \infty.$$

(iii) Thus, since $R > 0$ could have been any fixed positive number, we see that $|R_N(x)| \to 0$ at all $x \in \mathbb{R}$.

(iv) Thus, $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for all $x \in \mathbb{R}$.

7) $\sin(x)$ is analytic

We need to show that $R_N(x) = \sin(x) - \sum_{n=0}^{N} \frac{\sin^{(n)}(0)}{n!} x^n$ has $|R_N(x)| \to 0$ for all $x \in \mathbb{R}$ as $N \to \infty$.

(i) Notice that for every $n \ge 1$, $\left|\frac{d^n}{dx^n} \sin x\right| \le 1$ for all $x \in \mathbb{R}$.

(ii) By Taylor's Inequality, for any $x \in \mathbb{R}$,

$$0 \le |R_N(x)| \le \frac{|x|^N}{(N+1)!} \to 0 \text{ as } N \to \infty.$$

(iii) Thus, $\sin(x) = \sum_{n=0}^{\infty} \frac{\sin^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!}$ for all $x \in \mathbb{R}$.

8) This justifies our example from last time: $\cos(x)$ is also analytic and its Taylor series can be found via term-by-term differentiation:

$$\cos(x) = \sin'(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}.$$